Generalized parity transformations in the regularized Chern-Simons theory

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Abstract

We study renormalization effects in the Abelian Chern-Simons (CS) action. These effects can be non-trivial when the gauge field is coupled to dynamical matter, since the regularization of the UV divergences in the model forces the introduction of a parity even piece in the gauge field action. This changes the classical (odd) transformation properties of the pure CS action. This effect, already discussed for the case of a lattice regularization [1], is also present when the theory is defined in the continuum and, indeed, it is a manifestation of a more general 'anomalous' effect, since it happens for every regularization scheme. We explore the physical consequences of this anomaly. We also show that generalized, non local parity transformations can be defined in such a way that the regularized theory is odd, and that those transformations tend to the usual ones when the cutoff is removed. These generalized transformations play a role that is tantamount to the deformed symmetry corresponding to Ginsparg-Wilson fermions [2] (in an even number of spacetime dimensions).

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1 Introduction

It is a well-known fact that the regularization of the UV infinities of a quantum field theory may break some of the symmetries of the underlying classical theory. When this breaking is unavoidable, i.e., when it cannot be escaped just by choosing a suitable regularization, one has an 'anomaly' in the corresponding symmetry [3]. These anomalies have very important consequences, ranging from the constraints for model-building that follow from requiring anomaly cancellation, to the exact solution of $1+1$ dimensional models by performing decoupling transformations in a path integral.

In the present letter, we shall study an anomalous effect that occurs for a CS theory coupled to a matter field in $2+1$ dimensions. This effect amounts to a change in the classical behaviour of the action, which originally is purely odd under parity, to the 'mixed' behaviour of a sum of two terms with opposite parities. In fact, the general situation regarding this effect may be thought of as the replacement of the CS action by a Maxwell Chern Simons (MCS) like action. The use of a MCS action [4] at intermediate steps, considering the 'CS limit' at the end of the calculation, is of course a well known and extensively used procedure. We are here, however, considering it as a regularization, and studying it from the point of view of the symmetries of the quantum theory.

The replacement of the CS action by the MCS one affects, as we shall see, the structure of the renormalized effective action in certain models in such a way that, when the regulator is removed, a non vanishing anomalous effect remains. The case of the pure CS action minimally coupled to a Dirac field is dealt with in some detail, since this is perhaps the most natural system where this phenomenon shows up. On the other hand, this is a model that has been extensively studied because of its many interesting properties regarding, for example, the proper definition of relativistic anyon field operators.

The breaking of the classical parity odd behaviour of the system was analyzed for the lattice theory in [1], where it was shown that this breaking will happen for *any* sensible definition of the lattice CS action. It was also suggested in [1] that the lattice theory may, perhaps, verify a Ginsparg-Wilson [5] like relation. Indeed, the situation is, in more than one aspect, similar to the breaking of chiral symmetry on the lattice. For the Dirac operator in odd dimensions, the corresponding Ginsparg-Wilson relation and their related generalized parity transformations have been constructed on the

lattice [6] and in the continuum [7]. Moreover, it has also been shown [6] how those properties can be understood by dimensional reduction from even dimensions. The existence of a Ginsparg-Wilson like relation for the CS action would mean that, although the naive classical transformation properties of the gauge field are spoiled, there could exist a more subtle transformation involving the lattice operator, generalizing the parity odd nature of its continuum version. In the case of Ginsparg-Wilson fermions, the generalized chiral transformations are the ones discovered by Lüscher [2]. In this article, we show that a similar phenomenon occurs here for the parity transformations of the gauge field in the regularized CS action, and we apply it to the derivation of some consequences.

The organization of this paper is as follows: In section 2, we consider a Chern-Simons gauge field coupled to a Dirac field, in the continuum. We show that the regularization procedure naturally leads one to consider a Maxwell-Chern-Simons (MCS) theory rather than a pure CS one, hence breaking the classical odd transformation properties of the Chern-Simons field under parity. There is, however, a remnant of the classical behaviour which is manifested by the existence of a generalized parity transformation under which the MCS action is still odd. This is the content of section 3, which presents the symmetry transformations associated with the Ginsparg-Wilson like relation suggested in [1]. In section 4 we show that essentially the same symmetry holds true for the lattice Chern-Simons theory. In section 5, we apply the generalized symmetry to the derivation of some general relations, valid both for the continuum and lattice versions of the system, and present our conclusions.

2 Continuum theory

In order to explore the possible renormalization effects for a dynamical Chern-Simons field, we shall consider in this section a model consisting of a Dirac field minimally coupled to an Abelian Chern-Simons gauge field. The generating functional of complete Green's functions is:

$$\mathcal{Z}[j_\mu, \bar\eta, \eta] \;=\; \int [\mathcal{D}A_\mu] \mathcal{D}\bar\psi \mathcal{D}\psi \; \exp\left\{ i \int d^3x [\mathcal{L} \,+\, j^\mu A_\mu \,+\, \bar\eta \psi \,+\, \bar\psi \eta] \right\} \tag{1}$$

where

$$\mathcal{L} \;=\; \mathcal{L}_F \,+\, \mathcal{L}_{CS} \tag{2}$$

$$\mathcal{L}_F \;=\; \bar{\psi}(i\;\partial\!\!\!/ - e\;A\!\!\!/ - m)\psi\;, \tag{3}$$

and

$$\mathcal{L}_{CS} \;=\; \frac{\kappa}{2}\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda \tag{4}$$

is the Chern-Simons Lagrangian. In our conventions, the fermionic fields and κ have the dimensions of a mass, while the gauge field and e of a $(\text{mass})^{1/2}$. $[\mathcal{D}A_\mu]$ denotes the gauge field functional integration measure, including gauge fixing factors.

The need for a regularization is then clear, from the evaluation of the superficial degree of divergence $\omega(G)$ of a proper diagram G in the perturbative expansion of the generating functional (1):

$$\omega(G) \;=\; 3 - E_F - E_B \tag{5}$$

where E_F and E_B denote the number of external fermionic and bosonic lines in G, respectively. This counting corresponds to a *renormalizable* theory, and hence it leaves room for the existence of primitively divergent diagrams. Those are the vacuum polarization function ($\omega = 1$), the fermion self-energy ($\omega = 1$), and the vertex function ($\omega = 0$). A simple and convenient gauge invariant regularization scheme for a theory like this is, of course, the Pauli-Villars method. In this case, it amounts to replacing $\mathcal{L}$ by a 'regularized Lagrangian' $\mathcal{L}^{reg}$, defined by:

$$\mathcal{L}^{reg} \;=\; \mathcal{L}_F^{reg} + \mathcal{L}_{CS}^{reg} \tag{6}$$

where, by following the Pauli-Villars method, the fermion and gauge field Lagrangians have to be treated differently. For the fermionic Lagrangian one has to include extra regulator fields that improve the high momentum behaviour of the fermionic loops. In $2+1$ dimensions, and for the model we are considering, just one bosonic regulator $\bar{\phi}, \phi$ is sufficient to render all the fermionic loops convergent:

$$\mathcal{L}_F^{reg} \;=\; \bar{\psi}(i\;\partial\!\!\!/ - e\;A\!\!\!/ - m)\psi + \bar{\phi}(i\;\partial\!\!\!/ - e\;A\!\!\!/ - \Lambda)\phi \tag{7}$$

where Λ is a mass, proportional to the cutoff of the theory. Regarding the gauge field, the situation is slightly different, and we consider it now in more detail. To that end, we shall deal with the (unregularized) part of $\mathcal{Z}$ which depends on A_μ. It is clear that this object may be written as follows:

$$\mathcal{Z}_A[J] \;=\; \int \mathcal{D}A_\mu \, \exp\left\{ i \int d^3x [\mathcal{L}_{CS}(A) \,+\, J^\mu A_\mu] \right\}\;, \tag{8}$$

where J^μ denotes the full current to which A_μ is coupled, namely,

$$J^\mu \;=\; j^\mu \;-\; e\bar{\psi}\gamma^\mu\psi \;-\; e\bar{\phi}\gamma^\mu\phi \; . \tag{9}$$

The Pauli-Villars method [8], when applied to the gauge field A_μ, requires the introduction of one *massive* regulator field, B_μ, identically coupled to the current, and with a similar Lagrangian. The mass of B_μ is also proportional to the cutoff Λ. The regularized version of (8) is then:

$$\mathcal{Z}_A^{reg}[J] \;=\; \int \mathcal{D}A_\mu \, \mathcal{D}B_\mu \; \exp\left\{ i \int d^3x [\mathcal{L}_{CS}^{reg}(A,B) \;+\; J^\mu (A_\mu + B_\mu)] \right\} \; , \tag{10}$$

where we introduced $\mathcal{L}_{CS}^{reg}$, the 'regularized Chern-Simons Lagrangian', which is defined by:

$$\mathcal{L}_{CS}^{reg}(A,B) \;=\; \mathcal{L}_{CS}(A) \;-\; \mathcal{L}_{CS}(B) \;+\; \frac{1}{2} M^2 B_\mu B^\mu \; . \tag{11}$$

The idea behind the introduction of the massive field B_μ is to improve the large momentum behaviour of the loop integrals which contain a gauge field propagator without changing the low momentum behaviour. This is more clearly seen if B_μ is integrated out in (11), what can be done easily because the integral is quadratic. One defines a new field $\mathcal{A}_\mu = A_\mu + B_\mu$, changes variables from A_μ and B_μ to $\mathcal{A}_\mu$ and B_μ, and then integrates out B_μ to obtain:

$$\mathcal{Z}_A^{reg}[J] \;=\; \int \mathcal{D}\mathcal{A}_\mu \; \exp\left\{ i \int d^3x [-\frac{\xi^2}{4} \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} \;+\; \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} \mathcal{A}_\mu \partial_\nu \mathcal{A}_\lambda + J^\mu \mathcal{A}_\mu] \right\} \; , \tag{12}$$

which has a CS form, with $\xi = \frac{\kappa}{M}$ and $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$. This clearly shows that the introduction of the regulator improves the large momentum behaviour, since the MCS propagator goes like k^{-2} for large $|k|$, and on the other hand the pure CS action is recovered when $\xi \to 0$ ($M \to \infty$).

It is interesting to realize that the MCS theory *is* a Pauli-Villars regularized version of the CS theory. On the other hand, it has been known since a long time ago that it is convenient to evaluate observables in the pure CS theory by starting from the MCS action, and then to take the $\xi \to 0$ limit at the end of the calculation. This approach takes care of many possible sources of divergences when dealing with the pure CS action coupled to matter. For

example, the celebrated relation between magnetic field $B = \epsilon_{jk}\partial_j A_k$ and charge $\rho = J^0$

$$\kappa\, B(x) \;=\; -\rho(x) \tag{13}$$

of the CS theory is transformed, by the addition of a Maxwell term, into

$$-\,\xi^2 \Delta B(x) + \frac{\kappa^2}{\xi^2} B(x) \;=\; -\frac{\kappa}{\xi^2}\,\rho(x)\;. \tag{14}$$

In particular, for a static point-like source, the magnetic flux becomes

$$B(x) \;\propto\; K_0(Mr) \tag{15}$$

rather than a δ function.

The unusual fact here is that the regularized theory is a sensible physical model, devoided of the unphysical poles usually introduced by the Pauli-Villars regularization, when one deals with the regularization of more standard theories. The reason for this is that this regularization always adds an extra pole, and the requirement to improve large momentum behaviour demands the residue at the pole to be minus the one at the physical singularity. If there is a physical pole, then the regulator necessarily introduces an unphysical particle; however, for the pure CS gauge field the physical particle is missing, and thus the regulator can be chosen to correspond to a physical pole while improving the UV behaviour of the propagator.

The fact that the purely odd behaviour of S_{CS} is lost is evident, since the Maxwell action is parity even.

It is worth remarking that, in spite of the fact that the B_μ field has an explicit mass term, the regularization is gauge invariant. This is so because the gauge transformations in the regulated theory are defined by:

$$A_\mu \;\to\; A_\mu + \partial_\mu \omega \;\;,\;\; B_\mu \;\to\; B_\mu \tag{16}$$

which imply $\mathcal{A}_\mu \to \mathcal{A}_\mu + \partial_\mu \omega$. This shows the consistency of the assumption that B_μ does not change under gauge transformations, since then the regulated field $\mathcal{A}_\mu = A_\mu + B_\mu$ transforms in the same way as A_μ.

3 Generalized parity transformations

We shall study here the definition of the parity transformations, both for the cases of the standard pure CS action, and for the regularized (MCS) case.

The latter covers of course both the regularized CS theory and a theory defined *a priori* by a MCS action.

A parity transformation in $2+1$ dimensions is usually defined as a reflection along only *one* of the spatial coordinates, since a spatial inversion $\vec{x} \to -\vec{x}$ is, for a planar system, equivalent to a rotation (the Jacobian of the coordinate transformation is equal to $+1$). Thus,

$$x_\mu \;\to\; x^P_\mu \;\;:\;\; x^P_0 \;=\; x_0 \;,\; x^P_1 \;=\; -x_1 \;\;,\;\; x^P_2 = x_2 \tag{17}$$

is a possible definition of a parity transformation of the coordinates. The standard, unregularized CS action is odd under these transformations:

$$x_\mu \;\to\; x^P_\mu \quad A_\mu(x) \;\to\; A^P_\mu(x^P) \;=\; \frac{\partial x^P_\mu}{\partial x_\nu} A_\nu(x) \;, \tag{18}$$

since

$$\begin{aligned} S_{CS}[A] \;&=\; \int d^3x \, \frac{\kappa}{2} \, \epsilon^{\mu\nu\lambda} A_\mu(x) \frac{\partial}{\partial x_\nu} A_\lambda(x) \\ =\; -\int d^3x^P \, \frac{\kappa}{2} \, \epsilon^{\mu\nu\lambda} A^P_\mu(x^P) & \frac{\partial}{\partial x^P_\nu} A^P_\lambda(x^P) \;=\; -S_{CS}[A^P] \;. \end{aligned} \tag{19}$$

This action is also odd under time inversion. Thus we can also use a more symmetric expression for the transformations, introduced in [1], where 'parity' is defined by the full spacetime inversion:

$$x \;\to\; x^I \qquad x^I_\mu \;=\; -x_\mu \;, \tag{20}$$

which may be thought of as a time inversion composed with a spatial rotation in π. The I transformation has the notational convenience that it has an identical effect on all the coordinates. Everything we shall do for the I transformations has of course an immediate analogy for the P transformations.

In order to study the structure of these transformations and their generalizations, it is convenient to use a more abstract notation. For example, the vector field $A_\mu(x)$ and its Fourier transform $\tilde{A}_\mu(p)$ will be regarded as the coordinate and momentum representations, respectively, of some abstract vector field $|A_\mu\rangle$ in a Hilbert space:

$$A_\mu(x) \;=\; \langle x|A_\mu\rangle \;\;,\;\; \tilde{A}_\mu(p) \;=\; \langle p|A_\mu\rangle \;. \tag{21}$$

Then, for the I transformation acting on A_μ, we have:

$$\begin{aligned}\langle x^I | A^I_\mu \rangle \; &= \; -\langle x | A_\mu \rangle \\ \langle x | A^I_\mu \rangle \; &= \; -\langle x^I | A_\mu \rangle \; = \; -\langle x | \mathcal{I} | A_\mu \rangle \end{aligned} \tag{22}$$

where $\mathcal{I}$ denotes the operator that performs the I transformation on the coordinates. Its matrix elements are then

$$\langle x | \mathcal{I} | y \rangle \; = \; \langle x^I | y \rangle \; = \; \langle x | y^I \rangle \; = \; \delta^{(3)}(x+y) \; . \tag{23}$$

Then the inversion operator $\mathbf{I}$, when acting on A_μ is given by:

$$\mathbf{I} | A_\mu \rangle \; = \; | A^I_\mu \rangle \; = \; -\,\mathcal{I}\,| A_\mu \rangle \; , \tag{24}$$

or

$$\mathbf{I} \; = \; -\mathcal{I} \; . \tag{25}$$

With these conventions, and working now with Euclidean spacetime conventions, the CS action is written as:

$$S_{CS}[A] \; = \; \frac{\kappa}{2}\, \langle A_\mu | R_{\mu\nu} | A_\nu \rangle \tag{26}$$

with $R_{\mu\nu} \equiv i\epsilon_{\mu\lambda\nu}\partial_\lambda$, and its quality of being odd under $\mathbf{I}$ follows from:

$$S_{CS}[A^I] \; = \; \frac{\kappa}{2}\, \langle A^I_\mu | R_{\mu\nu} | A^I_\nu \rangle \; = \; \frac{\kappa}{2}\, \langle A_\mu | \, \mathcal{I}\, R_{\mu\nu}\, \mathcal{I}\, | A_\nu \rangle \; = \; -S_{CS}[A] \; , \tag{27}$$

which is tantamount to

$$\mathbf{I} R \mathbf{I} \; = \; -R \;\; \Leftrightarrow \;\; \{\mathbf{I}, R\} \; = \; 0 \, , \tag{28}$$

where the anticommutativity is derived by using also the obviously satisfied relation $\mathbf{I}^2 = 1$.

Let us now turn to the MCS action which, in Euclidean spacetime and with the above conventions can be written as

$$\begin{aligned} S_{MCS}[A] \; &= \; \frac{1}{2} \langle A_\mu | \left[\kappa R_{\mu\nu} \, + \, \frac{\kappa^2}{M^2} (-\partial^2) \delta^{\perp}_{\mu\nu} \right] | A_\nu \rangle \\ &\equiv \; \frac{\kappa}{2}\, \langle A_\mu | \tilde{R}_{\mu\nu} | A_\nu \rangle \end{aligned} \tag{29}$$

where $\delta^{\perp}_{\mu\nu} = \delta_{\mu\nu} - \dfrac{\partial_\mu \partial_\nu}{\partial^2}$ and

$$\tilde{R} \;=\; R(\,1 - \frac{\kappa}{M^2}\,R\,)\;. \tag{30}$$

The generalized inversion transformations $\tilde{\mathbf{I}}$ may be found by defining a general linear transformation for $|A_\mu\rangle$,

$$|A_\mu\rangle \to |A^I_\mu\rangle \;=\; \tilde{\mathbf{I}}|A_\mu\rangle \;=\; -\mathcal{I}\,(f\delta^{\perp}_{\mu\nu} \;-\; gR_{\mu\nu})|A_\nu\rangle \tag{31}$$

(with $\mathcal{I}$ as defined in (23)), and imposing the condition:

$$\tilde{\mathbf{I}}\,\tilde{R}_{\mu\nu}\,\tilde{\mathbf{I}} \;=\; -\tilde{R}_{\mu\nu}\;. \tag{32}$$

The scalar functions f and g are easily shown to be

$$\begin{aligned} f &= \left(1 - \xi^2 \frac{\partial^2}{M^2}\right)^{-\frac{1}{2}} \\ g &= \frac{\kappa}{M^2} \left(1 - \xi^2 \frac{\partial^2}{M^2}\right)^{-\frac{1}{2}} . \end{aligned} \tag{33}$$

The transformation law for the longitudinal part of the gauge field is arbitrary, since it is not determined by the equation (32). We choose, for simplicity,

$$\tilde{\mathbf{I}} | A_\mu \rangle = -\mathcal{I} \, (f \delta_{\mu\nu} - g R_{\mu\nu}) | A_\nu \rangle . \tag{34}$$

It is straightforward to check that the transformations defined by these coefficient functions indeed verify (32), by direct substitution. However, it is perhaps more instructive to realize that the generalized inversions defined by (31) can also be written as:

$$\tilde{\mathbf{I}} = -\mathcal{I} \sqrt{\frac{1 - \xi \frac{1}{M} R}{1 + \xi \frac{1}{M} R}} \tag{35}$$

where the property (32) is more explicit. We can also use (35) to show that $\{\tilde{\mathbf{I}}, \tilde{R}_{\mu\nu}\} = 0$, since

$$\begin{aligned} \tilde{\mathbf{I}}^2 &= \mathcal{I} \sqrt{\frac{1 - \xi \frac{1}{M} R}{1 + \xi \frac{1}{M} R}} \mathcal{I} \sqrt{\frac{1 - \xi \frac{1}{M} R}{1 + \xi \frac{1}{M} R}} \\ &= \sqrt{\frac{1 + \xi \frac{1}{M} R}{1 - \xi \frac{1}{M} R}} \sqrt{\frac{1 - \xi \frac{1}{M} R}{1 + \xi \frac{1}{M} R}} = 1 . \end{aligned} \tag{36}$$

We conclude this section by pointing out that the introduction of the Maxwell term to regulate the theory has a non-trivial effect on the renormalized theory. Indeed, the two-particle scattering amplitude including one-loop effects has a local repulsive interaction which survives even when taking the 'anyon limit' $\xi \to 0$ [9].

4 Lattice theory

We briefly discuss here the meaning of the symmetry presented in section 3 from the point of view of the lattice CS theory. To that end, we review the result presented in [1]. In this work, the authors show that given the CS action:

$$S_{CS} \;=\; \sum_{x,y} A_\mu(x) G_{\mu\nu}(x-y) A_\nu(y) \tag{37}$$

and assuming that it is local on the lattice, gauge invariant, and odd under parity, then (37) is not integrable. However, relaxing the last condition, the more general form of a local gauge invariant action in three dimensions includes a lattice Maxwell term

$$S_M \;=\; \sum_{x,y} A_\mu(x) M_{\mu\nu}(x-y) A_\nu(y) \tag{38}$$

where

$$M_{\mu\nu}(x-y) = -\Box\, \delta_{\mu\nu} + d_\mu \hat{d}_\nu \;. \tag{39}$$

In this equation $\Box = \sum_{\mu=0}^{2} d_\mu \hat{d}_\mu$ is the Laplacian in three dimensions. The forward and backward difference operators are given by $d_\mu f(x) = f(x+\hat{\mu}) - f(x) = (s_\mu - 1) f(x)$ and $\hat{d}_\mu f(x) = f(x) - f(x-\hat{\mu})(1 - s_\mu^{-1}) f(x)$ respectively, where s_μ is the forward translation operator.

It was shown in reference [1] that the only gauge invariant way to regularize the CS action is to add a parity even term such as the Maxwell term (39). The regularization of the extra zeroes in the CS action is due to the fact that the Maxwell term opens up a gap for them. Thus its introduction avoids one of the undesired features of the lattice CS action, at the price of destroying the odd behaviour of the pure CS action. However, we can show that, as in the continuum case, there is a generalized symmetry that is in fact preserved. That the same symmetry of Section 3 holds on the lattice, can be seen for example, from the Fourier version of (37) and (38), which is formally identical to its continuum counterpart, except from the different momentum ranges.

The lattice Fourier transformation of the gauge field A_μ is given by

$$A_\mu(x) = \int_{\mathcal{B}} \frac{d^3p}{(2\pi)^3} e^{-ipx} e^{-i\frac{p_\mu}{2}} \tilde{A}_\mu(p) \;, \tag{40}$$

where the lattice the integration over momenta is restricted to the Brillouin zone $\mathcal{B}$. Therefore the Fourier transformation of the Chern-Simons action is

$$S_{CS} = \int_{\mathcal{B}} \frac{d^3p}{(2\pi)^3} \tilde{A}_\mu(p) \tilde{G}_{\mu\nu}(p) \tilde{A}_\nu(-p) \tag{41}$$

with $\tilde{G}_{\mu\nu}(p) = e^{-i\frac{p_\mu}{2}} G_{\mu\nu}(p) e^{i\frac{p_\nu}{2}}$. It was shown in [1] that by requiring locality on the lattice, parity oddness and gauge invariance, the kernel $\tilde{G}_{\mu\nu}(p)$ must be of the form:

$$\tilde{G}_{\mu\nu}(p) = i\epsilon_{\mu\rho\nu} \hat{p}_\rho h(p) \tag{42}$$

with $\hat{p}_\rho = -2i \sin \frac{p_\rho}{2}$, and $h(p)$ an even analytic function of p.

As for the Maxwell term, its Fourier transformation is

$$S_M = \frac{1}{e^2} \int_{\mathcal{B}} \frac{d^3p}{(2\pi)^3} \tilde{A}_\mu(p) \tilde{M}_{\mu\nu}(p) \tilde{A}_\nu(-p) \tag{43}$$

being $\tilde{M}_{\mu\nu}(p) = e^{-i\frac{p_\mu}{2}} M_{\mu\nu}(p) e^{i\frac{p_\nu}{2}} = -\hat{p}^2 \delta_{\mu\nu} + \hat{p}_\mu \hat{p}_\nu$.

Using equations (41) and (43) it is simple to check that the Maxwell Chern-Simons action can be written as

$$S = \int_{\mathcal{B}} \frac{d^3p}{(2\pi)^3} \tilde{A}_\mu(p) \Gamma_{\mu\nu}(p) \tilde{A}_\nu(-p) \tag{44}$$

where

$$\Gamma_{\mu\nu}(p) = f(p) \delta^{\perp}_{\mu\nu}(p) + i g(p) Q_{\mu\nu}(p) \tag{45}$$

$$f(p) = \frac{4}{e^2} \sum_{\alpha=0}^{2} \sin^2 \frac{p_\alpha}{2} \tag{46}$$

$$g(p) = 2h(p) \sqrt{\sum_{\alpha=0}^{2} \sin^2 \frac{p_\alpha}{2}} \tag{47}$$

$$\delta^{\perp}_{\mu\nu} = \delta_{\mu\nu} - \frac{\sin(\frac{p_\mu}{2}) \sin(\frac{p_\nu}{2})}{\sum_{\alpha=0}^{2} \sin^2(\frac{p_\alpha}{2})} \tag{48}$$

$$Q_{\mu\nu} = \frac{1}{\sqrt{-\hat{p}^2}} \epsilon_{\mu\alpha\nu} \hat{p}_\alpha \tag{49}$$

We can now find the generalized parity transformations following the same steps as in Section 3. The result for the gauge field is

$$A^I_\mu(p^I) = i \frac{f(p)}{\sqrt{f^2(p) + g^2(p)}} Q_{\mu\rho} A_\rho(p) - \frac{g(p)}{\sqrt{f^2(p) + g^2(p)}} \delta^{\perp}_{\mu\rho} A_\rho(p) \tag{50}$$

Notice that when $f(p) \to 0$, i.e., when the action becomes the Chern-Simons action, we obtain the usual parity transformation $A^I_\mu(x^I) = -A_\mu(x)$. Therefore , this is the new parity transformation under which the Chern-Simons action is still odd but the kernel is integrable, and whose existence was suggested in reference [1].

Thus, we conclude that we can use the generalized inversion or parity transformation (50) as a substitute for the usual versions of those discrete symmetries. Indeed, as we shall see in the next section, the generalized symmetries are more useful than the usual ones when one tries to derive non-perturbative relations, like index theorems.

5 Conclusions

Let us derive some consequences from the generalized inversion (or parity) symmetry. We first note that one may use the (algebraic) relations involving the $\mathbf{I}$ operator and the kinetic operator $\tilde{R}_{\mu\nu}$ to construct an index theorem. Namely, we consider the quantity ω, the trace (in functional space and Lorentz indices) of $\mathbf{I}$:

$$\omega(\mathbf{I}) \;=\; \mathrm{Tr}(\mathbf{I}) \; . \tag{51}$$

Defining the (normalized) eigenvectors of R:

$$R_{\mu\nu}\phi^{(n)}_\nu(x) \;=\; \lambda_n \, \phi^{(n)}_\mu \tag{52}$$

the anticommutativity of R and $\mathbf{I}$ implies that the generalized inversion pairs eigenvectors of opposite λ_n. Then all except the $\lambda_n = 0$ states cancel out in the trace, when it is evaluated on the basis of the $\phi^{(n)}$:

$$\omega(\mathbf{I}) \;=\; \sum_{\lambda_n=0} \langle n|\mathbf{I}|n\rangle \;=\; n_+ - n_- \; , \tag{53}$$

where $n_\pm$ denote the number of zero modes of positive and negative parity, respectively. On the other hand, these zero modes verify

$$\epsilon_{\mu\nu\lambda}\partial_\nu\phi^{(0)}_\lambda(x) \;=\; 0 \tag{54}$$

and this means of course that $\phi^{(0)}_\lambda$ is locally a 'pure gauge' vector field $\phi^{(0)}_\lambda \;=\; \partial_\lambda\omega$. The number of independent (normalizable) zero modes will

thus be identical to the number of independent holonomies in the spacetime manifold where the action is defined. This, of course, has been derived in the context of the formal, unregularized CS action. Assuming now that one evaluates the trace of the standard inversion operator in the lattice theory, we immediately run into trouble, since there are spurious zero modes of the CS action that cancel out any non-vanishing contribution to the index theorem. The situation is of course analogous to the case of using the naive Dirac action for massless fermions on the lattice. However, having the generalized inversion symmetry, we may consider the trace of $\tilde{\mathbf{I}}$, which anticommutes with the lattice MCS operator $\tilde{R}$, and the non-zero modes are still paired. The problem with the spurious zero modes is avoided, because now there is a gap for these modes at the edges of the Brillouin zone.

Let us now show that the introduction of a parity even, Maxwell-like term in the action is not a special feature of the Pauli-Villars method, but rather a common feature of any sensible regularization implementable at the Lagrangian level. With full generality, a general regularized action S^{reg} can be written as

$$S^{reg}[\mathcal{A}] \;=\; S_{PV}[\mathcal{A}] \,+\, S_{PC}[\mathcal{A}] \tag{55}$$

where S_{PV} and S_{PC} denote parity violating and conserving terms, respectively. Obviously, the MCS action is a particular case of this general form, and it does regulate the gauge field propagator. Of course, one may think of more general actions corresponding to higher derivative versions of the MCS theory. For example,

$$S^{reg}[\mathcal{A}] \;=\; \frac{\kappa}{2}\langle A_\mu|u(\frac{\partial^2}{M^2})R_{\mu\nu}|A_\nu\rangle \,+\, \frac{1}{4}\langle F_{\mu\nu}|v(\frac{\partial^2}{M^2})|F_{\mu\nu}\rangle \;, \tag{56}$$

where u and v are functions chosen in order to obtain the desired behaviour for the resulting propagator. A crucial observation is that, to preserve gauge invariance, one cannot set $v \equiv 0$. The reason is that, for $v = 0$ we would have to require u to grow fast enough for large values of its argument, since the propagator would have a behaviour: $\sim [ku^{-1}(\frac{k^2}{M^2})]^{-1}$. If the manifold where the action is defined is such that it allows for large gauge transformations, like in finite temperature [10], this would reduce the symmetry group, by imposing extra constraints on the gauge transformation parameter. Indeed, an u which grows fast with momenta implies the existence of higher derivatives in the parity breaking piece of the action. This means that the

gauge variation of the action will involve higher derivatives of the gauge field parameter, which produce new constraints under the requirement of large gauge invariance (for example, by integrating by parts the gauge variation).

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